A brief introduction to rough path theory

粗糙路径理论简介

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Abstract. Rough path theory was originally developed by Terry Lyons in his seminal paper [Lyo98] *to understand differential systems with highly oscillatory-driven signals. In particular, it provides a fresh (if not evolutionary) methodology for stochastic differential systems (even beyond the framework of semimartingales where the classical Itô calculus works.) In this manuscript, we briefly introduce the ideas behind the theory and give some guidance and references on the study of its applications.*

1. Motivations and Basics

Rough path theory, roughly speaking, is a set of ideas and tools which allows a detailed analysis of irregular signals on non-linear systems. After two decades of rapid development, it has grown into a mature and widely applicable mathematical theory, which in particular offers a pathwise methodology for stochastic analysis. In [Hai13], Martin Hairer used rough path theory to solve the KPZ equation. Moreover, he was awarded the Fields Medal in 2014 for the invention of the theory of regularity structures, a major non-trivial extension of rough path theory. It has now grown into an essentially complete solution theory for general singular, subcritical semilinear (and quasilinear) stochastic partial differential equations.

In this manuscript, we aim to give a brief (and very incomplete) overview of rough path theory. We will introduce the integration and differential equation theory of rough paths, some consistency results and links between rough paths and stochastic processes (beyond semimartingales). A good complementary to this manuscript is the lecture of Weijun Xu in a summer school in Shanghai, which is available on Bilibili. Peter Friz also gave a great series of lectures at ETH in the Spring of 2020 (Videos available online). We encourage interested readers to read [CLL07] to begin with the study of the theory. Some other standard references are [FH14], [FV10] and [LQ02]. In particular, the comments in [FH14] have mentioned essentially every important work in this subject up to 2020. Besides, some introductory lecture notes that are much easier to read, e.g. [All21] and [Gen21], can be found online. If time permits and the readers are interested, we may update further manuscripts on some applications (e.g. in data science) or the theory of regularity structures.

To motivate the study of rough paths, we first look at a well-known ill-posed consequence of Itô calculus. Consider an SDE of the form

$$dY_t = f_0(Y_t)\,dt + f(Y_t)\,dB_t \tag{1}$$

where B is a standard Brownian motion. The solution map $S : B \mapsto Y$, known as the Itô map, lacks in general continuity, no matter what norm one used to equip the space of realisations of B. Indeed, one can first show the following negative result:

Proposition 1.1 ([FH14] Proposition 1.1)**.** *There exists no separable Banach space $\mathcal{B} \subset C([0,1])$ with the following properties:*

- *Sample paths of Brownian motions lie in $\mathcal{B}$ almost surely.*
- *The map $(f,g) \mapsto \int_0 f(t)dg(t)$ defined on smooth functions extends to a continuous map from $\mathcal{B} \times \mathcal{B}$ into the space of continuous functions on $[0,1]$.*

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Proof. See [FH14] Exercise 5.7. □

Let $B := (B^1, B^2)$ be a two-dimensional Brownian motion, the map $B \mapsto \int_0^{\cdot} B_t^1 \mathrm{d}B_t^2$ is exactly the Itô map associated to the SDE $\mathrm{d}Y_t = \mathrm{diag}(1, Y_t^1)\mathrm{d}B_t$. This map, which requires very little, already lacks continuity by the Proposition above. In this sense, solving Itô-SDEs is analytically ill-posed.
Rough paths theory, however, extends the continuity of the solution map to Brownian paths by taking into consideration also the iterated Itô-integrals of the driven signal against itself. In this sense, the solution map can be viewed as a continuous function of the path and iterated integrals under a suitable topology.
Generally, if already well-defined, the sequence of all iterated integrals is called the signature of a path, which will be introduced in the subsection below. Otherwise, one can enhance irregular paths with objects which mimic iterated integrals. In the next few sections, we will see that such objects are essentially sufficient to develop an integration and differential equation theory for irregular signals in a consistent way, which in particular restores the continuity of solution maps.

1.1. Signature of a regular path

In this Subsection, we introduce two fundamental properties of the iterated integrals of a regular path against itself, which will be the algebraic and analytic properties to postulate in the definition of a rough path. To quote Lyons et al. in the introduction of Chapter 3 of [CLL07]: *"The core idea of the theory of rough paths is to consider the signature, rather than the path, as the fundamental object"*.

Definition 1.2. Let J be a compact interval, V a Banach space, $T(V)$ the tensor algebra over V and $X : J \to V$ a regular (e.g. smooth or continuous with finite variation) path. The signature of X, denoted by $\mathbf{X}_J = \left(1, X_J^1, X_J^2, \ldots\right) \in T(V)$, is the collection of iterated (Riemann-Stieltjes) integrals

$$X_J^n = \int_{\substack{u_1 < \cdots < u_n \\ u_1, \ldots, u_n \in J}} \mathrm{d}X_{u_1} \otimes \cdots \otimes \mathrm{d}X_{u_n}.$$

We may also use the notation $S(X)$ to denote the signature of X.

Proposition 1.3 (Chen). *Let $X : [0, s] \longrightarrow V$, $Y : [s, t] \longrightarrow V$ be two smooth (or continuous of finite variation) paths and $X * Y : [0, t] \longrightarrow V$ their concatenation:*

$$(X * Y)_u = \begin{cases} X_u & \text{if } u \in [0, s] \\ X_s + Y_u - Y_s & \text{if } u \in [s, t]. \end{cases}$$

Then

$$S(X * Y) = S(X) \otimes S(Y) \tag{2}$$

where $\otimes$ denotes the canonical multiplication in the tensor algebra $T(V)$.

Proof. Set $Z = X * Y$ and $S(Z) = \left(1, Z^1_{0,t}, Z^2_{0,t}, \ldots\right)$. For arbitrary $n \geq 1$, we have:

$$\begin{aligned} Z^n_{0,t} &:= \int \cdots \int_{0<u_1<\ldots<u_n<t} dZ_{u_1} \otimes \ldots \otimes Z_{u_n} \\ &= \sum_{k=0}^{n} \int \cdots \int_{0<u_1<\ldots<u_k<s<u_{k+1}<\ldots<u_n<t} dZ_{u_1} \otimes \ldots \otimes dZ_{u_n} \\ &= \sum_{k=0}^{n} \int \cdots \int_{0<u_1<\ldots<u_k<s} dX_{u_1} \otimes \ldots \otimes dX_{u_k} \otimes \int \cdots \int_{s<u_{k+1}<\ldots<u_n<t} \otimes \ldots \otimes dY_{u_n} \\ &= \sum_{k=0}^{n} X^k_{0,s} \otimes Y^{n-k}_{s,t} \end{aligned}$$

where the third equality is by Fubini's theorem. Hence, $S(Z) = S(X) \otimes S(Y)$. □

Eq.(2) is called Chen's relation. It encodes the important algebraic property of signatures and asserts that the map S is a homomorphism. Another important and inspiring, while analytical, property of the signature is that it allows recovering integrals $\int f(X_r)\mathrm{d}X_r$ for n-time differentiable functions f, via a level-n compensated Riemann-Stieltjes sum. For simplicity, we consider here only paths in $\mathbb{R}$, for which we have the obvious equivalence:

$$X^n_{[s,t]} = \frac{1}{n!}(X_t - X_s)^n \tag{3}$$

Proposition 1.4. *Let $X : [0,t] \longrightarrow \mathbb{R}$ be a smooth (or continuous bounded variation) path and $f \in C^{n+1}(\mathbb{R})$, we have the following equality:*

$$\int_0^t f(X_r)\mathrm{d}X_r = \lim_{|\mathcal{P}|\to 0} \sum_{[t_i,t_{i+1}]\in\mathcal{P}} \sum_{k=0}^{n} f^{(k)}(X_{t_i}) \cdot X^{k+1}_{[t_i,t_{i+1}]}. \tag{4}$$

In particular, for $n = 0$ we recover the definition of the usual Riemman-Stieltjes integral.

Proof. First, by an $(n+1)$-order Taylor expansion and Langrange's mean value theorem, we have for any $r \in [t_i, t_{i+1}]$:

$$f(X_r) = \sum_{k=0}^{n} f^{(k)}(X_{t_i}) \cdot \frac{1}{k!}(X_r - X_{t_i})^k + f^{(n+1)}(X_\xi) \cdot \frac{1}{(n+1)!}(X_r - X_{t_i})^{n+1} \tag{5}$$

for some $\xi \in [t_i, r]$.
Then, one has:

$$\begin{aligned} \text{LHS} &= \lim_{|\mathcal{P}|\to 0} \sum_{[t_i,t_{i+1}]\in\mathcal{P}} \int_{t_i}^{t_{i+1}} f(X_r)\mathrm{d}X_r \\ &= \lim_{|\mathcal{P}|\to 0} \sum_{[t_i,t_{i+1}]\in\mathcal{P}} \int_{t_i}^{t_{i+1}} \sum_{k=0}^{n} f^{(k)}(X_{t_i}) \cdot \frac{1}{k!}(X_r - X_{t_i})^k \mathrm{d}X_r \\ &\quad + \lim_{|\mathcal{P}|\to 0} \sum_{[t_i,t_{i+1}]\in\mathcal{P}} \int_{t_i}^{t_{i+1}} f^{(n+1)}(X_\xi) \cdot \frac{1}{(n+1)!}(X_r - X_{t_i})^{n+1}\mathrm{d}X_r \end{aligned}$$

where the first term is exactly the RHS by eq.(3). Moreover, denoting the second term by Ξ, we have:

$$\begin{aligned}\Xi &\lesssim \lim_{|\mathcal{P}|\to 0} \sum_{[t_i,t_{i+1}]\in\mathcal{P}} \sup_{r\in[t_i,t_{i+1}]} (X_r - X_{t_i})^{n+1} \cdot |X|_{TV;[t_i,t_{i+1}]} \\ &\lesssim \lim_{|\mathcal{P}|\to 0} \sum_{[t_i,t_{i+1}]\in\mathcal{P}} (|X|_{(n+1)-var;[t_i,t_{i+1}]})^{n+1} = 0\end{aligned}$$

where the first line follows from the property of Riemann-Stieltjes integrals and the second line is due to the regularity assumption on X. □

Indeed, as we shall see later, the compensated Riemann-Stieltjes sum in eq.(4) is exactly how we construct the integration theory for rough paths.

Remark 1.5. The study of signatures is sometimes also considered within the framework of the theory of rough paths. Although we will not delve any deeper, it is worth mentioning that the signature method has had many interesting applications in data science and machine learning (particularly in the recognition of hand-written Chinese characters), see e.g. [Lyo14], [Gra13], [CK16] and [LM22].

1.2. Rough paths

Now we may finally introduce the definition of rough paths, which in essence mimics the signature of a regular path.

Definition 1.6 (Rough path)**.** Let $\alpha \le 1$, V a Banach space, Δ_T the 2-simplex over the interval $[0,T]$ and $T^{(\lfloor \frac{1}{\alpha} \rfloor)}$ the truncated tensor algebra up to level $\lfloor \frac{1}{\alpha} \rfloor$. We say that $\mathbf{X} = \left(1, \mathbb{X}^1, \ldots, \mathbb{X}^{\lfloor \frac{1}{\alpha} \rfloor}\right)$: $\Delta_T \to T^{(\lfloor \frac{1}{\alpha} \rfloor)}$ is an α-Hölder rough path if for all $0 \le s \le u \le t \le T$ we have:

$$\mathbf{X}_{s,t} = \mathbf{X}_{s,u} \otimes \mathbf{X}_{u,t}, \tag{6}$$

and for all $(s,t) \in \Delta_T$ and all $i = 1, \cdots, \lfloor \frac{1}{\alpha} \rfloor$ we have

$$\left\|\mathbb{X}^i_{s,t}\right\| \lesssim (t-s)^{\alpha i}. \tag{7}$$

The space of α-Hölder rough paths over $[0,T]$ on V is denoted $\mathbf{\Omega}^\alpha([0,T],V)$, equipped with the following α-Hölder metric:

$$d^\alpha(\mathbf{X},\mathbf{Y}) := \sum_{1\le i\le \lfloor \frac{1}{\alpha} \rfloor} \left\|\mathbb{X}^i - \mathbb{Y}^i\right\|_{(i\alpha)-\text{ Höl}} .$$

We may drop the contents of the parentheses whenever the context is clear.

It is clear that eq.(6) is the imitation of eq.(2). In particular, we have the additivity of $\mathbb{X}^1$, which means it is indeed the parametrisation of a path. As we will see in the next Section, the analytical requirement for graded Hölder regularity in eq.(7) is essentially what is needed to construct an integration theory against rough paths. It might seem strange why we require only a definition for the "signature type" object up to a finite level of the tensor algebra. Indeed, an α-Hölder rough path extends automatically to a proper "signature" on the whole tensor algebra, even uniquely in some sense, see e.g. [LV07] and Theorem 3.7 in [CLL07].

Example 1.7. Not surprisingly, one can show that the truncated up to level-$\lfloor \frac{1}{\alpha} \rfloor$ signature of a smooth path is a α-Hölder rough path. We call it the canonical α-Hölder rough path lift.

Remark 1.8. Note that for a rough path $\mathbf{X}$, the parameterization of the underlying path is already given by $\mathbb{X}^1$, by the first level of eq.(6). For the simplicity of notations, we will use $\mathbf{X} := (X, \mathbb{X})$ to denote a rough path for $\alpha \in (\frac{1}{3}, \frac{1}{2}]$ in the following. This is the case that we will focus on in the rest of the manuscript. For the discussion of rough paths of lower regularities, see e.g. [FH14] or [FV10]. In our case, eq.(6) essentially becomes the additivity of $\mathbb{X}^1$ and

$$\mathbb{X}^2_{s,t} - \mathbb{X}^2_{s,u} - \mathbb{X}^2_{u,t} = \mathbb{X}^1_{s,u} \otimes \mathbb{X}^1_{u,t}, \tag{8}$$

and eq.(7) becomes

$$\|\mathbb{X}^1\|_{\alpha},\ \|\mathbb{X}^2\|_{2\alpha} < \infty. \tag{9}$$

We note that Terry Lyons and Nicolas Victoir proved in [LV07] that given an α-Hölder path X for $\alpha \in (\frac{1}{3}, \frac{1}{2}]$, there always exists an α-Hölder rough path lift $\mathbf{X} = (X, \mathbb{X})$, albeit not canonically. Another interesting question is that given such an X, up to which point is its rough path lift $\mathbf{X}$ determined by eq.(8) and eq.(9). Indeed, we have the following immediate result:

Proposition 1.9. *Let X be as above and $\mathbf{X}$, $\tilde{\mathbf{X}}$ two α-Höler rough paths with $\mathbb{X}^1_{s,t} = \tilde{\mathbb{X}}^1_{s,t} = X_t - X_s$. Then $\tilde{\mathbb{X}}^2_{s,t} - \mathbb{X}^2_{s,t} = G_{s,t}$ for some 2α-Hölder continuous function G. Conversely, for any 2α-Hölder continuous function F, $(\mathbb{X}^1_{s,t}, \mathbb{X}^2_{s,t} + G_{s,t})$ defines again an α-Hölder rough path.*

Yet another important component of a solid integration theory is an understanding of change in variables, e.g. a chain rule or Itô's formula. We now give two ways of postulating a preliminary chain rule to a rough path.

Definition 1.10 (Geometric rough path)**.** For $\alpha \leq 1$, We define the space of geometric α-Hölder rough paths by $G\Omega^{\alpha}([0,T], V) \subset \Omega^{\alpha}([0,T], V)$ as the closure of canonical rough path lifts of all smooth paths on $[0, T]$, under the α-Hölder topology given by the metric in Definition 1.6.

Definition 1.11 (Weakly geometric rough path)**.** An α-Hölder rough path is called weakly geometric if it takes value in the step-$\lfloor \frac{1}{\alpha} \rfloor$ nilpotent group $G^{\lfloor \frac{1}{\alpha} \rfloor}(V) \subset T^{\lfloor \frac{1}{\alpha} \rfloor}(V)$. We denote the space of weakly geometric α-Hölder rough paths by $WG\Omega^{\alpha}([0,T], V)$. In particular, for $V = \mathbb{R}^n$ and $\alpha \in (\frac{1}{3}, \frac{1}{2}]$, this is equivalent to

$$\mathrm{Sym}\left(\mathbb{X}^2_{s,t}\right) = \frac{1}{2}\mathbb{X}^1_{s,t} \otimes \mathbb{X}^1_{s,t}. \tag{10}$$

Remark 1.12. Indeed, one can directly show that the canonical rough path lifts of smooth paths are weakly geometric by the Riemann-Stieltjes integration by parts. Moreover, one can show the following strict inclusion:

$$WG\Omega^{\alpha}([0,T], V) \subsetneq G\Omega^{\alpha}([0,T], V).$$

Besides, some more relations between them are discussed in [FH14] Chapter 2.

One of the most important examples of rough paths is the enhanced Brownian motion. Indeed, almost all Brownian paths can be lifted to rough paths via iterated Itô- or Stratonovich integrals and we will later see that the integration theory against those rough paths is consistent with the classical stochastic calculus. Indeed, one has:

Proposition 1.13 (Enhanced Brownian motion)**.** *Let B be a d-dimensional Brownian motion and define*

$$\mathbb{B}^{\mathrm{Ito}}_{s,t} := \int_s^t B_{s,r} \otimes dB_r \quad \textit{and} \quad \mathbb{B}^{\mathrm{Strat}}_{s,t} := \int_s^t B_{s,r} \otimes dB_r.$$

For any $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right)$, with probability one, we have

$$\mathbf{B}^{\text{Ito}} := \left(B, \mathbb{B}^{\text{Ito}}\right) \in \Omega^{\alpha}\left([0,T], \mathbb{R}^d\right) \textit{ and}$$

$$\mathbf{B}^{\text{Strat}} := \left(B, \mathbb{B}^{\text{Strat}}\right) \in G\Omega^{\alpha}\left([0,T], \mathbb{R}^d\right).$$

Proof. See [FH14] Proposition 3.4 and Proposition 3.5. □

2. Integration theory of rough paths

From now on, we focus on the simplest but non-trivial case $\alpha \in (\frac{1}{3}, \frac{1}{2}]$. For rough paths of lower regularities, we refer to [BDFT21] Chapter 2 for a compact introduction. See also [FH14] Section 2.4 and [Gub10].
In this section, we give an overview of the integration theory of rough paths. We cite most results from [FH14] Chapter 4. The approach that we present originally contributed to Massimiliano Gubinelli in [Gub04]. It has been generalised to arbitrary α in [Gub10]. Nevertheless, we mention that the first integration theory of rough paths was introduced in [Lyo98], which was limited to integrations on one-forms against geometric rough paths.
The intuition behind Gubinelli's approach is that if "Y looks like X at small scales", then one can construct $\int Y \mathrm{d}X$ from $\int X \mathrm{d}X$, which is encoded in the definition of a rough path. In the first Subsection, we explain what "Y looks like X at small scales" means, while in the second Subsection, we construct the rough integral.

2.1. Controlled rough paths

We begin with a definition and some examples for some intuitions.

Definition 2.1 (Controlled paths)**.** Let $\alpha \in (1/3, 1/2]$, and let V, W be Banach spaces. For a path $X \in C^{\alpha}([0,T], V)$, we say that a path $Y \in C^{\alpha}([0,T], W)$ is controlled by X if there exists a one-form $Y' \in C^{\alpha}([0,T], \mathcal{L}(V,W))$ such that the remainder $R^Y : \Delta_T \to W$ is defined by the relation

$$Y_{s,t} = Y'_s\left(X_{s,t}\right) + R^Y_{s,t}, \tag{11}$$

is such that $||R^Y||_{2\alpha} < \infty$. The space of paths controlled by X is denoted by $\mathcal{D}^{2\alpha}_X([0,T], W)$, whose elements are the pairs (Y, Y') defined as above. For such elements (Y, Y') we call Y' the Gubinelli derivative of Y with respect to X
We endow the space $\mathcal{D}^{2\alpha}_X$ with the seminorm

$$\|(Y, Y')\|_{X,2\alpha} := \|Y'\|_{\alpha} + \left\|R^Y\right\|_{2\alpha}. \tag{12}$$

Remark 2.2. The intuition for the pair (Y, Y') is that Y looks like X locally, where the local similarity is described by Y' with a local error of order 2α. We note that the Gubinelli derivative is not unique when Y is "too smooth" (cf. [FH14] Remark 4.7). Moreover, we will see in Section 4 that paths controlled by X can also be viewed as rough paths, which provides some necessary consistency for the theory.

Now we give some important examples of controlled paths.

Example 2.3 (Functions of the reference path)**.** Let $F : V \to \mathcal{L}(V, W)$ be a C^2_b function and X as above. Then $F(X)$ is controlled by X by the Gubinelli derivative $DF(X)$. The proof is immediate or see [FH14] Lemma 4.1. In particular, X is itself a controlled path with Gubinelli derivative Id.

Example 2.4 (Functions of controlled paths)**.** More generally, let X be as above, (Y,Y') a controlled path in W and F a proper C_b^2 function. Then one can define a controlled rough path $F(Y,Y'):=(F(Y),F(Y)')$ via

$$F(Y)_t := F(Y_t), \quad F(Y)'_t := DF(Y_t)\,Y'_t.$$

The proof is straightforward or see [FH14] Lemma 7.3 for a proof. Moreover, as a consequence of the chain rule, we have the consistency that $(F\circ G)(Y,Y') = F(G(Y,Y'))$. We note that such constructions can be generalised for any $\mathrm{Lip}(\gamma-1)$ function F for $2<\gamma<3$.

Example 2.5 (Composition of controlled paths)**.** Let X be as above, (Y,Y') a controlled path in $\mathcal{L}(W,\bar{W})$ and (Z,Z') a controlled path in W. Then one can define a controlled rough path $(YZ, Y'Z+YZ')$ in $\bar{W}$, where we used the canonical isomorphism

$$\mathcal{L}(V,\mathcal{L}(W,\bar{W}))\times W \cong \mathcal{L}(V,\bar{W})$$

in writing $Y'Z$. Indeed, for the remainder we have

$$\begin{aligned}\left|R_{s,t}\right| &:= \left|Y_tZ_t - Y_sZ_s - Y'_sZ_sX_{s,t} - Y_sZ'_sX_{s,t}\right|\\ &= \left|Y_tZ_t - Y_sZ_s - Y_{s,t}Z_s - Y_sZ_{s,t} + R^Y_{s,t}Z_s + Y_sR^Z_{s,t}\right|\\ &\lesssim \left|Y_{s,t}Z_{s,t} + R^Y_{s,t}Z_s + Y_sR^Z_{s,t}\right| \sim |t-s|^{2\alpha}.\end{aligned}$$

Example 2.6 (Transitivity of controlled paths)**.** Similarly, let X be as above, (Y,Y') controlled by X and (Z,Z') controlled by Y. Then, Z is also controlled by X via the Gubinelli derivative $Z'Y'$. Indeed, we have $Z = Z'Y'X + Z'R^Y_X + R^Z_Y$, where the remainder is clearly 2α-Hölder by assumptions.

2.2. Rough Integration

We are now ready to construct the integral of a path controlled by X against an associated rough path $\mathbf{X}=(X,\mathbb{X})$, which mimics essentially the (n=1) compensated Riemann-Stieltjes sum as in Proposition 1.4. Let $Y\in C^\alpha([0,T],\mathcal{L}(V,W))$ be a controlled one-form by X with Gubinelli derivative Y', we define the rough integral by

$$\int_0^T (Y,Y')\mathrm{d}\mathbf{X} := \lim_{|\mathcal{P}|\to 0}\sum_{[s,t]\in\mathcal{P}}\left(Y_sX_{s,t} + Y'_s\mathbb{X}_{s,t}\right) \tag{13}$$

where we used the canonical injection $\mathcal{L}(V,\mathcal{L}(V,W)) \hookrightarrow \mathcal{L}(V\otimes V, W)$ in writing $Y'_s\mathbb{X}_{s,t}$. Recall that when Y and X are α-Hölder continuous for some $\alpha<\frac{1}{2}$, the Rieman-Stieltjes integral $\lim_{|\mathcal{P}|\to 0}\sum_{[s,t]\in\mathcal{P}} Y_sX_{s,t}$ does not converge in general. However, this is overcome by the compensated one via the following theorem:

Theorem 2.7. *Let* $\mathbf{X}$ *and* (Y,Y') *be as above, the rough integral in eq.(13) is well defined. Moreover, we have* $(\int_0^\cdot (Y,Y')\mathrm{d}\mathbf{X}, Y)\in\mathscr{D}_X^{2\alpha}(W)$.

It is a consequence of the following sewing lemma, which originates from [Gub04]. For this, we introduce the space $C_2^{\alpha,\beta}([0,T],W)$ of functions Ξ from the 2 -simplex $\{(s,t): 0\le s\le t\le T\}$ into W such that $\Xi_{t,t}=0$ and such that

$$\|\Xi\|_{\alpha,\beta} = \|\Xi\|_\alpha + \|\delta\Xi\|_\beta < \infty,$$

where $\|\Xi\|_\alpha = \sup_{s<t}\frac{|\Xi_{s,t}|}{|t-s|^\alpha}$ as usual, and also

$$\delta\Xi_{s,u,t} := \Xi_{s,t} - \Xi_{s,u} - \Xi_{u,t}, \quad \|\delta\Xi\|_\beta := \sup_{s<u<t}\frac{\left|\delta\Xi_{s,u,t}\right|}{|t-s|^\beta}.$$

Lemma 2.8 (**Sewing lemma; Gubinelli, 2004**). *Let α and β be such that $0 < \alpha \le 1 < \beta$. Then, there exists a unique linear map $\mathcal{I} : C_2^{\alpha,\beta}([0,T],W) \to C^\alpha([0,T],W)$ such that $(\mathcal{I}\Xi)_0 = 0$ and*

$$\left|(\mathcal{I}\Xi)_{s,t} - \Xi_{s,t}\right| \le C|t-s|^\beta. \tag{14}$$

where C only depends on β and $\|\delta\Xi\|_\beta$. $\mathcal{I}$ is called the sewing map. (The α-Hölder norm of $\mathcal{I}\Xi$ also depends on $\|\Xi\|_\alpha$ and hence on $\|\Xi\|_{\alpha,\beta}$.)

Proof. Uniqueness is immediate. Fix an Ξ and assume I and $\bar{I}$ both satisfy eq.(14). It follows that $I - \bar{I}$ satisfies $(I-\bar{I})_0 = 0$ and $\left|(I-\bar{I})_{s,t}\right| = \left|(I-\bar{I})_t - (I-\bar{I})_s\right| \lesssim |t-s|^\beta$. Since $\beta > 1$ by assumption, we conclude that $I - \bar{I}$ vanishes identically.
For existence, we construct $I := \mathcal{I}\Xi$ via its increment $I_{s,t}$ for any s,t by some successive refinement on dyadic intervals. Wlog, we assume $T = 1$. First, let $[s,t] = 2^{-k}[l, l+1]$ for some $k \ge 0$ and $l \in \{0, ..., 2^k - 1\}$ an elementary dyadic interval. Let $\mathcal{P}_n$ be the level-n dyadic partition of $[s,t]$, which contains 2^n intervals, each of length 2^{-n}, starting with the trivial partition $\mathcal{P}_0 = \{[s,t]\}$. Define $I^0_{s,t} = \Xi_{s,t}$ and then the nth level approximation by

$$I^{n+1}_{s,t} := \sum_{[u,v]\in\mathcal{P}_{n+1}} \Xi_{u,v} = I^n_{s,t} - \sum_{[u,v]\in\mathcal{P}_n} \delta\Xi_{u,m,v},$$

where the second equality is straightforward. It then follows immediately from the definition of $\|\delta\Xi\|_\beta$ that

$$\left|I^{n+1}_{s,t} - I^n_{s,t}\right| \le 2^{n(1-\beta)}|t-s|^\beta \|\delta\Xi\|_\beta.$$

This means the sequence $I^n_{s,t}$ is Cauchy since $\beta > 1$ and the required eq.(14) is obtained by summing up the above bound. Moreover, for such $[s,t]$, we have $I_{s,t} = I_{s,\frac{(s+t)}{2}} + I_{\frac{(s+t)}{2},t}$ by taking limit of the obvious identity $I^{n+1}_{s,t} = I^n_{s,\frac{(s+t)}{2}} + I^n_{\frac{(s+t)}{2},t}$. Then we can extend this construction to general dyadic intervals $2^{-k}[l,m]$ by

$$I_{2^{-k}\ell, 2^{-k}m} = \sum_{j=\ell}^{m-1} I_{2^{-k}j, 2^{-k}(j+1)},$$

which is obviously additive for all dyadic numbers. Now we verify eq.(14) for them. For $s < t$ dyadic, we consider a partition $P = (\tau_i)$ of $[s,t]$ with elementary dyadic intervals such that no three intervals have the same length. Thanks to the additivity of I and eq.(14) for elementary dyadic intervals, we have:

$$\begin{aligned}
\left|I_{s,t} - \Xi_{s,t}\right| &= \left|\sum_{[u,v]\in P} (I_{u,v} - \Xi_{u,v}) - \left(\Xi_{s,t} - \sum_{[u,v]\in P} \Xi_{u,v}\right)\right| \\
&\lesssim \sum_{[u,v]\in P} |v-u|^\beta + \left|\Xi_{s,t} - \sum_{[u,v]\in P} \Xi_{u,v}\right| \\
&\le |t-s|^\beta + \sum_{i=0}^{\infty} \left|\delta\Xi_{s,\tau_i,\tau_{i+1}}\right| \lesssim |t-s|^\beta + \sum_{i=0}^{\infty} (\tau_{i+1} - s)^\beta,
\end{aligned} \tag{15}$$

where infinite sums are actually finite. Now denote by L the mesh of the partition. Assuming wlog that the lengths of intervals in P are monotone decreasing, we have for the latter term:

$$\sum_{i=0}^{\infty}(\tau_{i+1}-s)^{\beta} = \sum_{i=0}^{\infty}(\sum_{k=1}^{i}(\tau_{k+1}-\tau_k))^{\beta}$$
$$\leq \sum_{i=0}^{\infty}(\sum_{j=1}^{i-1}\frac{L}{2^j})^{\beta} \sim L^{\beta} \sim (t-s)^{\beta}.$$

Finally, we extend our constructions for dyadic numbers continuously to $[0,T]$. □

Now the existence of the rough integral in eq.(13) is a consequence of the sewing lemma.

Proof of Theorem 2.7. We set $\Xi_{s,t} := Y_s X_{s,t} + Y'_s \mathbb{X}_{s,t}$ By Chen's relation in eq.(8), a straightforward computation shows

$$\delta\Xi_{s,u,t} = -R^Y_{s,u}\left(X_{u,t}\right) - Y'_{s,u}\left(\mathbb{X}_{u,t}\right).$$

Then one has $\Xi \in C_2^{\alpha,3\alpha}([0,T],W)$ where $3\alpha > 1$ by the assumptions on $\mathbf{X}$ and (Y,Y'). Then, applying the sewing lemma to Ξ and taking limit of sums of eq.(14) along any sequence of partitions gives

$$\lim_{|\mathcal{P}|\to 0}\sum_{[s,t]\in\mathcal{P}}\left(Y_s X_{s,t} + Y'_s\mathbb{X}_{s,t}\right) = \mathcal{I}\Xi,$$

which means the rough integral is well-defined via eq.(13). Finally, that $(\mathcal{I}\Xi, Y) \in \mathcal{D}_X^{2\alpha}(W)$ follows directly from eq.(14). □

An understanding of the change of variables in rough integration will be presented in Section 5, provided some consistency between controlled paths and rough paths. Moreover, we note that some continuity results for rough integration can be found in [FH14] Theorem 4.10 and Theorem 4.17.

Example 2.9 (Consistency with stochastic analysis)**.** For an Itô Brownian rough path $B(\omega)$, the integral of $(Y(\omega), Y'(\omega)) \in \mathcal{D}^{2\alpha}_{B(\omega)}([0,T],W)$, the integral $\int(Y,Y')d\mathbb{B}^{Ito}$ exists with probability one, and moreover if (Y,Y') are adapted then this integral is equal to the classic Itô integral. This result is proved in [FH14] Section 5.1, and the corresponding result for Stratonovich is shown in Section 5.2. These results are significant as they show that rough integration is consistent with existing stochastic theory.

Example 2.10 (Perturbation of rough integration)**.** Let f be a 2α-Hölder continuous function and let $\mathbf{X} = (X,\mathbb{X})$ and $\overline{\mathbf{X}} = (X,\overline{\mathbb{X}})$ be two rough paths that differ by f (cf. Proposition 1.9). Let furthermore $(Y,Y') \in \mathcal{D}_X^{2\alpha}$. It follows immediately from eq.(13) that

$$\int_s^t (Y_r, Y'_r)\mathrm{d}\overline{\mathbf{X}}_r = \int_s^t (Y_r, Y'_r)\mathrm{d}\mathbf{X}_r + \int_s^t Y'_r\mathrm{d}f(r).$$

Here, the second term on the right-hand side is a simple Young integral, which is well-defined since $\alpha + 2\alpha > 1$ by assumption.

Similar to the constructions above, we can also construct the integral of a controlled path against a controlled path via the sewing lemma. In particular, as we will see in Section 4, associating a controlled path with its iterated integral in this sense allows us to translate it to a rough path.

Proposition 2.11. *Let* $\mathbf{X} := (X, \mathbb{X})$ *be an* α*-Hölder rough path,* $(Y, Y') \in \mathcal{D}_X^{2\alpha}(\mathcal{L}(W, \bar{W}))$ *and* $(Z, Z') \in \mathcal{D}_X^{2\alpha}(W)$*. Then we have a well-defined integral via:*

$$\int (Y, Y')\mathrm{d}(Z, Z') := \lim_{|\mathcal{P}|\to 0} \sum_{[s,t]\in\mathcal{P}} (Y_s Z_{s,t} + Y_s' Z_s' \mathbb{X}_{s,t}). \tag{16}$$

where we identify $Y_s' Z_s'$ *as a linear map via the canonical isomorphism* $\mathcal{L}(V, \mathcal{L}(W, \bar{W})) \times \mathcal{L}(V, W) \cong \mathcal{L}(V \otimes V, \bar{W})$*. Moreover, the integral is again controlled by* $\mathbf{X}$ *with Gubinelli derivative* YZ*.*

Proof. Same as the proof of Theorem 2.7, whereas we set $\Xi_{s,t} := Y_s Z_{s,t} + Y_s' Z_s' \mathbb{X}_{s,t}$. □

3. Solutions of Rough Differential Equations (RDEs)

In this Section, we give a meaning to differential equations driven by rough paths. Essentially, solutions are constructed in the space of controlled paths. Existence and uniqueness results will be cited. We mainly present results from [FH14] Chapter 8, where RDE solutions are understood in an equivalent manner as understood by Davie in [Dav08]. Nevertheless, we note that the original theory of solutions to RDE by Terry Lyons in [Lyo98], was described in a slightly different manner. For this, we also refer to [CLL07] Chapter 5 and [FH14] Section 8.8. Essentially, the two notions are of the same spirit, although solutions in the sense of Lyons were constructed in a context without controlled paths.

Definition 3.1. Let $\alpha \in (\frac{1}{3}, \frac{1}{2}]$, V, W Banach spaces and $\mathbf{X} \in \Omega^\alpha([0, T], V)$. For a C_b^2 function $f : W \to L(V, W)$, the controlled path $(Y, Y') \in \mathcal{D}_X^{2\alpha}([0, T], W)$ is a solution to the RDE

$$\mathrm{d}Y_t = f(Y_t)\,\mathrm{d}\mathbf{X}_t, \quad Y_0 = y_0 \in W \tag{17}$$

if for all $t \in [0, T]$ we have that Y satisfies the integral equality

$$Y_t = y_0 + \int_0^t (f(Y)_s, f(Y)_s')\mathrm{d}\mathbf{X}_s \tag{18}$$

where $(f(Y), f(Y)')$ is constructed from (Y, Y') as in Example 2.5.

Now we cite a result for the existence and uniqueness of solutions, which is essentially based on a rough version of Picard-Lindelöf iteration.

Theorem 3.2. *In the above setting, we let instead* $f \in C^3(W, L(V, W))$*. Then there exists a unique element* $(Y, Y') \in \mathcal{D}_X^{2\alpha}([0, T], W)$ *such that* (Y, Y') *is a solution to the RDE*

$$\mathrm{d}Y_t = f(Y_t)\,\mathrm{d}\mathbf{X}_t, \quad Y_0 = y_0 \in W$$

for times $t \in [0, T]$*. Furthermore, we have* $Y' = f(Y)$ *and for* f *that is also bounded, the solutions are global in time.*

Proof. See [FH14] Theorem 8.4. The proof uses Banach's fixed point theorem on the space of controlled paths, endowed with a proper norm. □

Remark 3.3 (Links to SDEs)**.** An important property of such solutions is that the corresponding solution map is continuous, see e.g. [FH14] Theorem 8.5. Moreover, in the case of enhanced Brownian motion (cf. Proposition 1.13), the notion of solutions to RDEs is consistent with the notion of solutions to classical SDEs, see e.g. [FH14] Theorem 9.1. Thus, the theory of RDEs indeed resolves the ill-posedness of the theory of stochastic differential equations proposed in Section 1. Based on this, rough paths theory (in particular for enhanced Brownian motion), provides also another perspective for many other aspects of stochastic analysis, e.g. approximation of solutions to SDEs, stochastic flows, support theorems and asymptotic behaviours. For this, we refer to [FH14] Chapter 9 and [FV10] Chapter 17, 18 and 19.

Remark 3.4 (RDEs with drift)**.** In many situations, we are also interested in solutions to the eq.(17) with an additional drift:

$$\mathrm{d}Y_t = f_0(Y,t)\mathrm{d}t + f(Y,t)\mathrm{d}\mathbf{X}_t. \tag{19}$$

On the one hand, it is possible to recast eq.(19) in the form of eq.(17) by writing it as an RDE for $\hat{Y}_t = (Y_t, t)$ driven by $\hat{\mathbf{X}}_t = (\hat{X}, \hat{\mathbb{X}})$ where $\hat{X} = (X_t, t)$ and $\hat{\mathbb{X}}$ is given by $\mathbb{X}$ and the "remaining cross integrals" of X_t and t, given by usual Riemann-Stieltjes integration. However, it is possible to exploit the structure of such RDEs with drift to obtain somewhat better bounds on the solutions. For this, we refer to [FV10] Chapter 12.

4. Controlled rough paths revisited: consistency

The theory developed in the previous sections has an obvious drawback, that is, the integrands and integrators are objects of different types. In particular, the notion of controlled paths is ad hoc, depending explicitly on the chosen reference paths. However, this can be resolved by viewing controlled paths as rough paths via its iterated rough integral. This also induces some other consistency results for the rough integration and differential equation theory. In particular, it helps to understand the change of variables in rough integration. Most results in this section are cited from different chapters of [FH14].
Throughout this section, let $\mathbf{X}$, (Y, Y'), (Z, Z') etc be the same as in the previous section. We begin by constructing a canonical map $\mathcal{D}_X^{2\alpha}(W) \hookrightarrow \Omega^{\alpha}(W)$.

Proposition 4.1 (Controlled paths as rough paths)**.** *Let* $\mathbf{X} := (X, \mathbb{X})$ *be an* α*-Hölder rough path,* (Y, Y') *a path controlled by* $\mathbf{X}$*. Then* $\mathbf{Y} := (Y, \mathbb{Y})$ *is also an* α*-Hölder rough path, where*

$$\mathbb{Y}_{s,t} := \int_s^t Y_{s,r} \otimes dY_r = \lim_{|\mathcal{P}|\to 0} \sum_{[u,v]\in\mathcal{P}} (Y_{s,u} \otimes Y_{u,v} + Y'_u \otimes Y'_u \mathbb{X}_{u,v}).$$

In particular, by setting $(Y, Y') := (X, Id)$ *we recover* $\mathbf{Y} = \mathbf{X}$*. Note that the integral is exactly in the sense of Proposition 2.11.*

Proof. Set $\Xi_{u,v} := Y_{s,u} \otimes Y_{u,v} + Y'_u \otimes Y'_u \mathbb{X}_{u,v}$. We have $\|\mathbb{Y}\|_{2\alpha} < \infty$ as an immediate consequence of $\left|\mathcal{I}(\Xi)_{s,t} - \Xi_{s,t}\right| \lesssim |t-s|^{3\alpha}$ from the sewing lemma and $\left\|\Xi_{s,t}\right\|_{2\alpha} < \infty$. Chen's relation is obvious from summing up the facts that $\delta\left(\int_s^t Y_r \otimes \mathrm{d}Y_r\right)_{s,u,t} = 0$ (abstract integration property) and that $\delta\left(\int_s^t Y_s \otimes \mathrm{d}Y_r\right)_{s,u,t} = -Y_{s,u} \otimes Y_{u,t}$. The recovery of $\mathbf{X}$ is just a sanity check by Chen's relation. □

In particular, the construction preserves the weakly geometric property:

Proposition 4.2 (Weakly geometricity is preserved)**.** *Let* $\mathbf{X}$*,* (Y, Y') *and* $\mathbf{Y}$ *be as above and* $\mathbf{X}$ *additionally weakly geometric in* $\mathbb{R}^d$*,* Y *in* $\mathbb{R}^n$*. Then* $\mathbf{Y}$ *is again weakly geometric.*

Proof. Set $\Xi_{u,v} := Y_{s,u} \otimes Y_{u,v} + Y'_u \otimes Y'_u \mathbb{X}_{u,v}$. By the weakly geometricity of $\mathbf{X}$ we have:

$$2\mathrm{Sym}(Y'_u \otimes Y'_u \mathbb{X}_{u,v}) = Y'_u \otimes Y'_u 2\mathrm{Sym}(\mathbb{X}_{u,v}) = Y'_u \otimes Y'_u (X_{u,v} \otimes X_{u,v}).$$

Thus, we have

$$
\begin{aligned}
\Xi^{i,j}_{u,v} + \Xi^{j,i}_{u,v} &= Y^i_{s,u} Y^j_{u,v} + Y^j_{s,u} Y^i_{u,v} + Y'^i_u X_{u,v} \cdot Y'^j_u X_{u,v} \\
&= Y^i_{s,u} Y^j_{u,v} + Y^j_{s,u} Y^i_{u,v} + Y^i_{u,v} \cdot Y^j_{u,v} + O\left(|v-u|^{3\alpha}\right) \\
&= Y^i_s Y^j_{u,v} + Y^j_s Y^i_{u,v} + (Y^i_v Y^j_v - Y^i_u Y^j_u) + O\left(|v-u|^{3\alpha}\right)
\end{aligned}
\tag{20}
$$

where the second equality is by the simple observation that Y^i is again controlled by X with Gubinelli derivative Y'^i for any component Y^i and the third equality is a simple computation. Now summing them up along any $\mathcal{P}$ and letting $|\mathcal{P}| \to 0$, we obtain as desired $\mathbb{Y}^{i,j}_{s,t} + \mathbb{Y}^{j,i}_{s,t} = Y^i_{s,t} Y^j_{s,t}$. □

Another important property of this embedding is that the integral of a controlled path against another controlled path (cf. Proposition 2.11) is the same as when viewing the integrator as a rough path and the integral in the sense of eq.(13), as in the following result:

Proposition 4.3 (Consistency). *Let* $\mathbf{X}$*,* (Y, Y') *and* $\mathbf{Y}$ *be as above. If* $(Z, Z') \in \mathcal{D}^{2\alpha}_Y$*, then we have the equality*

$$
\int_s^t (Z_r, Z'_r) \mathrm{d}\mathbf{Y}_r = \int_s^t (Z_r, Z'_r Y'_r)\ \mathrm{d}(Y_r, Y'_r)
$$

where $(Z_r, Z'_r Y'_r)$ *is a path controlled by* X *by Example 2.6. The left-hand side is understood in the sense of eq.(13), while the right-hand side in the sense of eq.(16).*

Proof. By definition, the second integral has local approximation:

$$
\begin{aligned}
\Xi_{u,v} &= Z_u Y_{u,v} + Z'_u Y'_u \mathbb{X}_{u,v} \\
&= Z_u Y_{u,v} + \tilde{Z}_u Y'_u Y'_u \mathbb{X}_{u,v}
\end{aligned}
$$

By definition of $\mathbb{Y}$, we have the local approximation estimate $\left|\mathbb{Y}_{s,t} - Y'_u Y'_u \mathbb{X}_{u,v}\right| \sim |t-s|^{3\alpha}$ so that the first integral has local approximation

$$
\begin{aligned}
\bar{\Xi}_{u,v} &= \bar{Z}_u Y_{u,v} + \tilde{Z}'_u \mathbb{Y}_{u,v} \\
&= \Xi_{u,v} + O\left(|t-s|^{3\alpha}\right).
\end{aligned}
\tag{21}
$$

Then, for the integrals on a fixed interval $[s, t]$, which are just the image of the local approximations under the sewing map in Lemma 2.8, one has

$$
\begin{aligned}
|\mathcal{I}\Xi_{s,t} - \mathcal{I}\bar{\Xi}_{s,t}| &\leq \lim_{|\mathcal{P}|\to 0} \sum_{[u,v]\in\mathcal{P}} \left(|\mathcal{I}\Xi_{u,v} - \Xi_{u,v}| + |\bar{\Xi}_{u,v} - \mathcal{I}\bar{\Xi}_{u,v}| + |\Xi_{u,v} - \bar{\Xi}_{u,v}|\right) \\
&\sim \lim_{|\mathcal{P}|\to 0} \sum_{[u,v]\in\mathcal{P}} (u-v)^{3\alpha} = 0
\end{aligned}
\tag{22}
$$

where "∼" is by the error estimates in the sewing lemma and eq.(21), and the final equality is since $3\alpha > 1$. □

The proposition above essentially uses the transitivity of controlled paths (cf. Example 2.6). Similarly, using the composition of controlled paths (cf. Example 2.5), one obtains an associativity property for rough integrations.

Proposition 4.4 (Associativity)**.** *Let* $\mathbf{X}$ *be as above and* $(Y, Y'), (K, K') \in \mathcal{D}_X^{2\alpha}$ *two controlled paths defined in proper Banach spaces such that the two integrals in eq.(23) exist. Moreover, let* $(Z, Z') := \left(\int_0^{\cdot} (K_u, K_u') \mathrm{d}\mathbf{X}_u, K\right) \in \mathcal{D}_X^{2\alpha}$ *be another controlled path by Theorem 2.7. Then*

$$\int_0 (Y_u, Y_u') \,\mathrm{d}(Z_u, Z_u') = \int_0 (Y_u K_u, Y_u' K_u + Y_u K_u') \mathrm{d}\mathbf{X}_u \tag{23}$$

where the integral on the left-hand side is in the sense of eq.(16) and the integral on the left-hand side is in the sense of eq.(13) and Example 2.5.

Proof. The proof is similar to the proof for Proposition 4.3. By writing down the local approximations for the two integrals, one sees that they differ by an $O(|t-s|^{3\alpha})$ term. Then the sewing map sews them to the same object by the same arguments as in eq.(22). □

Up to this point, we are able to understand every object that we have seen, particularly integrands in rough integrals and solutions to RDEs, in the space of rough paths. However, it still lacks an understanding of the change of variables, which will be introduced below.

5. An Ito type formula for rough integration

In classical stochastic calculus, the change of variables is understood via the Itô formula, in which the quadratic variation of a stochastic process is involved. Recall that the quadratic variation requires a probabilistic construction. In this subsection, we introduce a pathwise corresponding object which plays the same role in rough path theory. This object is called the bracket of a rough path.

Definition 5.1. Let $\mathbf{X} = (X, \mathbb{X}) \in \Omega^\alpha([0,T], V)$ for some $\alpha \in (\frac{1}{3}, \frac{1}{2}]$. The bracket of $\mathbf{X}$ is defined as the path $[\mathbf{X}] : [0,T] \to V \otimes V$ given by

$$[\mathbf{X}]_t := X_{0,t} \otimes X_{0,t} - 2\,\mathrm{Sym}\left(\mathbb{X}_{0,t}\right)$$

Remark 5.2. By Chen's relation, it is straightforward to verify that

$$[\mathbf{X}]_{s,t} := [\mathbf{X}]_t - [\mathbf{X}]_s = X_{s,t} \otimes X_{s,t} - 2\,\mathrm{Sym}\left(\mathbb{X}_{s,t}\right) \tag{24}$$

for all s, t. In particular, $[\mathbf{X}]$ is 2α- Hölder. Moreover, a rough path is weakly geometric if and only if its bracket is trivial. In other words, the bracket describes the non-geometric part of a rough path.

One derives easily from Proposition 1.13 that the bracket of the $\mathbf{B}^{\mathbf{Ito}}$ coincides with the quadratic variation of a Brownian motion. More generally, one has the following associativity property:

Proposition 5.3. *Let* $\mathbf{X} = (X, \mathbb{X})$ *be as above and* $(K, K') \in \mathcal{D}_X^{2\alpha}$*. Recall that* $(Z, Z') := \left(\int_0 (K_u, K_u') \,\mathrm{d}\mathbf{X}_u, K\right) \in \mathcal{D}_X^{2\alpha}$*. Let* $\mathbf{Z} = (Z, \mathbb{Z})$ *be its canonical rough path lift so that in particular the bracket* $[\mathbf{Z}]$ *of* $\mathbf{Z}$ *exists. Then*

$$[\mathbf{Z}] = \int_0 (K_u \otimes K_u) \,\mathrm{d}[\mathbf{X}]_u$$

where the integral on the right-hand side is a Young integral.

Proof. Since $[\mathbf{X}]$ is 2α-Hölder continuous, the right-hand side exists as a Young integral. Moreover, we have

$$\begin{aligned}[\mathbf{Z}]_{s,t} &= Z_{s,t} \otimes Z_{s,t} - 2\,\mathrm{Sym}\left(\mathbb{Z}_{s,t}\right) \\ &= \left(K_s X_{s,t} + K_s' \mathbb{X}_{s,t}\right) \otimes \left(K_s X_{s,t} + K_s' \mathbb{X}_{s,t}\right) - 2\left(Z_s' \otimes Z_s'\right) \mathrm{Sym}\left(\mathbb{X}_{s,t}\right) + O\left(|t-s|^{3\alpha}\right) \\ &= \left(K_s X_{s,t}\right) \otimes \left(K_s X_{s,t}\right) - 2\left(K_s \otimes K_s\right) \mathrm{Sym}\left(\mathbb{X}_{s,t}\right) + O\left(|t-s|^{3\alpha}\right) \\ &= \left(K_s \otimes K_s\right) [\mathbf{X}]_{s,t} + O\left(|t-s|^{3\alpha}\right).\end{aligned}$$

Taking $\lim_{|\pi|\to 0} \sum_{[s,t]\in\pi}$ on both sides, we obtain $[\mathbf{Z}]_T = \int_0^T \left(K_u \otimes K_u\right) \mathrm{d}[\mathbf{X}]_u$. □

Abusing notation slightly, one could rewrite the above result as

$$\left[\int_0^{\cdot} K_u \,\mathrm{d}\mathbf{X}_u\right]_t = \int_0^t K_u^2 \,\mathrm{d}[\mathbf{X}]_u,$$

which coincides with the quadratic variation of Itô integrals if we set $\mathbf{X} = \mathbf{B}^{\mathbf{Ito}}$.

We are now ready to prove the following rough Itô formula:

Theorem 5.4 (Rough Itô formula)**.** *Let* $\mathbf{X} = (X, \mathbb{X})$ *be as above and* $f \in C^3$*. Then*

$$f\left(X_T\right) = f\left(X_0\right) + \int_0^T \left(Df\left(X_u\right), D^2 f\left(X_u\right)\right) \mathrm{d}\mathbf{X}_u + \frac{1}{2}\int_0^T D^2 f\left(X_u\right) \mathrm{d}[\mathbf{X}]_u,$$

where the first integral on the right-hand side is a rough integral, and the second integral is a Young integral.

Proof. Since X is bounded, we assume without loss of generality that $f \in C_b^3$. We have:

$$f\left(X_t\right) - f\left(X_s\right) = Df\left(X_s\right) X_{s,t} + \frac{1}{2} D^2 f\left(X_s\right)\left(X_{s,t} \otimes X_{s,t}\right) + R_{s,t}$$

where

$$R_{s,t} := \int_0^1 \int_0^1 \left(D^2 f\left(X_s + r_1 r_2 X_{s,t}\right) - D^2 f\left(X_s\right)\right)\left(X_{s,t} \otimes X_{s,t}\right) r_1 \,\mathrm{d}r_2 \,\mathrm{d}r_1.$$

Note that

$$\left|R_{s,t}\right| \leq \|f\|_{C_b^3} \left|X_{s,t}\right|^3 \lesssim |t-s|^{3\alpha},$$

so that $\lim_{|\pi|\to 0} \sum_{[s,t]\in\pi} \left|R_{s,t}\right| = 0$. Since $D^2 f\left(X_s\right)$ is symmetric and by eq.(24), we have

$$f\left(X_t\right) - f\left(X_s\right) = \left(Df\left(X_s\right) X_{s,t} + D^2 f\left(X_s\right) \mathbb{X}_{s,t}\right) + \frac{1}{2} D^2 f\left(X_s\right) [\mathbf{X}]_{s,t} + R_{s,t}.$$

Taking $\lim_{|\pi|\to 0} \sum_{[s,t]\in\pi}$ on both sides, we deduce the result. □

Corollary 5.5 (Chain rule)**.** *Let* $\mathbf{X}$*,* f *be as above and moreover* $\mathbf{X}$ *weakly geometric, we have the following chain rule:*

$$f\left(X_T\right) = f\left(X_0\right) + \int_0^T \left(Df\left(X_u\right), D^2 f\left(X_u\right)\right) \mathrm{d}\mathbf{X}_u.$$

Proof. This is because the bracket of a weakly geometric rough path is 0. □

Remark 5.6. The above formulas are restricted to integrands that are functions of the underlying path. Indeed, as long as (Y, Y') and (Y', Y'') are controlled paths, one can replace them with the corresponding derivatives in the above formulas. See [FH14] Theorem 7.7 for a concrete description. Besides, see [BDFT21] Theorem 3.9 for an Itô formula for weakly geometric rough paths of lower regularity.

6. Conclusions and Comments

Recall that we are initially interested in studying the differential control system

$$\mathrm{d}Y_t = f(Y_t)\,\mathrm{d}X_t \tag{25}$$

where X is irregular, in the sense that it is less regular than $\frac{1}{2}$-Hölder. To start with, one needs to construct a well-defined integral $\int Y \mathrm{d}X$. However, the normal Riemann-Stieltjes sum does not in general converge by the low regularity of X. This was however resolved in Section 2 by postulating X with a second-order increment $\mathbb{X}$, viewing Y as a controlled path and then taking the compensated Riemann-Stieltjes sum as in 13. In Section 3 we saw that such systems, viewed as RDE, admit unique solutions for proper f via a Picard-Lindelöf iteration in the space of controlled paths. Section 4 then unifies the notion of controlled paths and rough paths in some sense and allows us to view rough integrals, and solutions to RDEs as rough paths. Section 5 then finally gives an understanding of the change of variables in rough integrals.

Throughout this manuscript, we see that rough path theory is particularly compatible with Itô (or Stratonovich-) Integral. Indeed, the connections with probabilistic objects go far beyond that. A lot of stochastic processes (e.g. certain Gaussian processes and Markov processes) can be naturally lifted to rough paths, which allows analysis of random systems driven by those signals, see e.g. [FV10]. Nevertheless, we note that the choice of rough path lifts does matter and one has to be careful. For instance, in the context of finance, the correct rough paths lift of a Brownian motion is the Itô enhancement in order to exclude arbitrage opportunities (since we recover the Black-Scholes model in this case).

In this manuscript, we only considered continuous paths. However, we note that cadlag rough paths and rough paths with jumps were also quite well-studied by P. Friz et al. In particular, A. Allan, C. Liu, D. Prömel et al were able to model financial objects in such settings.

In the theory of rough integration, the set of integrands is actually quite limited since controlled paths have to look like the reference paths locally. In the setting of stochastic analysis, people are sometimes interested in path-dependent integrands. For instance, the strategy of a stockholder might depend on the stock prices during a period of time. Indeed, Anna Ananova proved in her PhD thesis that such strategies can still be viewed as a controlled path in some slightly weaker sense, as long as the dependence is sufficiently regular in some sense. Her approach used the non-anticipative functional analysis developed by B. Dupire, R. Cont et al and ensures the existence of a rough integral.

A main drawback of this theory in the context of stochastic analysis, is that it does not give a general way to compute expectations. As far as the author knows, this is not (and not likely to be) solved even in the easiest non-semimartingale case, i.e. the case of fractional Brownian motions.

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